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1987 J. Phys. A: Math. Gen. 20 L193

(http://iopscience.iop.org/0305-4470/20/4/001)

J. Phys. A: Math. Gen. **20** (1987) L193–L194. Printed in the UK

LETTER TO THE EDITOR

$\nabla \times \boldsymbol{a} = k\boldsymbol{a}$ as a model of a force-free magnetic field

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Received 26 November 1986

Abstract. It has been claimed that the solutions to the vector differential equation $\nabla \times \boldsymbol{a} = k\boldsymbol{a}$ with k constant, considered as a model for force-free magnetic fields in plasma physics and in astrophysics, exhibit several mathematical inconsistencies. This claim is repudiated.

If the solutions to the equation

$$\nabla \times \boldsymbol{a} = k\boldsymbol{a} \tag{1}$$

are to be identified with physical magnetic fields then they must be invariant under (i) gauge transformations, (ii) parity and (iii) change of basis. Salingaros (1986) claims that solutions exist which do not exhibit such invariance and therefore rejects equation (1) as a model for a force-free magnetic field. His arguments are in fact fallacious as shown below.

(i)† If the solution to equation (1) is identified with a magnetic field $\boldsymbol{B}$ then $\boldsymbol{B}$ can be related directly to the vector potential $\boldsymbol{A}$ since

$$\nabla \times \boldsymbol{B} = k\boldsymbol{B} = k\nabla \times \boldsymbol{A} \Rightarrow \boldsymbol{B} = k\boldsymbol{A} + \nabla\phi$$

where ϕ is a scalar field. Since $\boldsymbol{B}$ is to be invariant under a gauge transformation $\boldsymbol{A}' = \boldsymbol{A} + \nabla\lambda$ the scalar field ϕ itself must transform as

$$\phi' = \phi - k\lambda + \text{constant}.$$

Salingaros treats ϕ as gauge invariant, this error leading to a false conclusion concerning the non-gauge invariance of the solutions of equation (1).

(ii) In equation (1) the constant k is a pseudoscalar, it changes sign under parity. With this observation the equation is in fact invariant under parity.

(iii) Salingaros derives the solution

$$\boldsymbol{a}(x^2, x^3) = [\psi, (1/k)\partial_3\psi, -(1/k)\partial_2\psi] \tag{2}$$

with

$$(\partial_2^2 + \partial_3^2 + k^2)\psi = 0$$

to equation (1), the coordinates being rectangular cartesian. He then shows that the *form* of the solution is invariant under translations and rotations but not under more general transformations. His conclusion that the equation is dependent on the choice

† Maheswaran (1986) has already made this point in a recently published letter.

of reference system is false. Covariance does *not* mean form invariance under a general coordinate transformation. The solution (2) is a special case of the solution

$$\boldsymbol{a} = \boldsymbol{\nabla} \times (\boldsymbol{c}\psi) + (1/k)\boldsymbol{\nabla} \times \boldsymbol{\nabla}(\boldsymbol{c}\psi)$$

where $\boldsymbol{c}$ is a constant vector field and ψ satisfies

$$\nabla^2\psi + k^2\psi = 0$$

first obtained by Chandrasekhar and Kendall (1957). To see this put $\boldsymbol{c} = \hat{\boldsymbol{\imath}}$ and choose ψ to be independent of x^1.

References

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